

(1, 2)-NULL BERTRAND CURVES IN MINKOWSKI SPACETIME

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Abstract. In this paper we study null Bertrand curves in R_1^4 under the assumption the curve has a Cartan frame. We show that if the derivative vectors of the null Cartan curve in R_1^4 is linearly independent, then this curve is not a Bertrand curve. Since then the already known notion of null Bertrand curves in R_1^4 occurs only if the derivative vectors of the curve is linearly dependent. We will introduce an idea of Bertrand curves and abiding by this idea we bring to light under which conditions a null Cartan curve in R_1^4 is a Bertrand curve.

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1 Introduction

Hiroo Matsuda ve Shinsuke Yoroze in their paper [3] introduced a new type of curves called special Frenet curves and proved that a special Frenet curve in R^n is not a Bertrand curve if $n \geq 4$. They also improved an idea of generalized Bertrand curve in R^4 . Particularly they characterized (1, 3)-Bertrand curves in R^4 and illustrate this type of curves with an example. Honda-Inoguchi [4] and Inoguchi-Lee [5] have done some work on a pair of null curves (C, \bar{C}) , called a null Bertrand pair and their relation with null helices in R_1^3 . For a null curve in the Minkowski n-space, Ferrandez, Gimenez and Lucas in [2] assumed the linear independence of the derivative vectors of the curve to get a unique Cartan frame for the curve. On the other hand in [7] Makoto Sakaki uttered that the assumption in the Theorem 2 in [2] can be lessened for a null curve to have a unique Cartan frame in R_1^n . In [1] A.Ceylan Çöken and Ünver Çiftçi studied null Bertrand Curves in Minkowski Spacetime.

Here we show that a null Cartan curve in R_1^4 is not a Bertrand curve if the derivative vectors of the curve is linearly independent and give a characterization of (1, 2)-Bertrand curves in R_1^4 .

2 Preliminaries

Let $c : I \longrightarrow R_1^n$, $n = m + 2$, be a null curve parametrized by the pseudo-arc parameter such that $\{c'(s), c''(s), \dots, c^{(n)}(s)\}$ is a basis of $T_{c(s)}R_1^n$ for all s . Then there exist only one Frenet frame satisfying the equations

$$\begin{aligned}
 L' &= W_1, \\
 N' &= k_1 W_1 + k_2 W_2, \\
 W_1' &= -k_1 L - N, \\
 W_2' &= -k_2 L + k_3 W_3, \\
 W_i' &= -k_i W_{i-1} + k_{i+1} W_{i+1} \quad i \in \{3, \dots, m-1\} \\
 W_m' &= -k_m W_{m-1},
 \end{aligned} \tag{2.1}$$

and verifying

- (i) For $2 \leq i \leq m-1$, $\{c', c'', \dots, c^{(i+2)}\}$ and $\{L, N, W_1, \dots, W_i\}$ have the same orientation.
- (ii) $\{L, N, W_1, \dots, W_m\}$ is positively oriented [2].

Assume that $c : I \rightarrow R_1^n$ be a null Cartan curve. Then for the Cartan curvatures of C , we obtain $k_2 < 0$, $k_i > 0$ for all $i \in \{3, \dots, m-1\}$, and $k_m > 0$ or $k_m < 0$ according to $\{c', c'', c^{(3)}, \dots, c^{(n)}\}$ is positively or negatively oriented, respectively [2].

Let (C, \overline{C}) be a pair of framed null Cartan curves in R_1^4 , with distinguished parameters s and \overline{s} respectively. This pair is said to be a null Bertrand pair if their principal normal vector fields are linearly dependent. The curve \overline{C} is called a Bertrand mate of C and vice versa. A framed null curve is said to be a null Bertrand curve if it admits a Bertrand mate. By this definition there exist a functional relation $\overline{s} = \varphi(s)$ for a null Bertrand pair (C, \overline{C}) such that $\overline{W}(\overline{s}) = \mp W(s)$, i.e., the normal lines coincide at their corresponding points [6].

3 Bertrand curves in R_1^4

Let C be a null Cartan curve in R_1^4 . After a straightforward computation we can write the following equation

$$\begin{pmatrix} c' \\ c'' \\ c^{(3)} \\ c^{(4)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k_1 & -1 & 0 & 0 \\ -k_1' & 0 & -2k_1 & -k_2 \end{pmatrix} \begin{pmatrix} L \\ N \\ W_1 \\ W_2 \end{pmatrix}$$

If we denote the above matrix by P , then $|P| = -k_2$. Here we choose the spacelike vector W_2 uniquely so that the pseudo-orthonormal basis $\{L, N, W_1, W_2\}$ is positively oriented. If we assume that $\{c^{(i)}\}_{1 \leq i \leq 4}$ is positively oriented, then we get $k_2 < 0$. Notice that if $k_2 = 0$, then $|P| = 0$ and this says that the set $\{c^{(i)}\}_{1 \leq i \leq 4}$ is linearly dependent.

Theorem 3.1 *No null Cartan curve C in R_1^4 is a Bertrand curve if $\{c^{(i)}\}_{1 \leq i \leq 4}$ is linearly independent.*

Proof: Let C be a Bertrand curve in R_1^4 and \overline{C} a Bertrand mate of C . \overline{C} is distinct from C . Let the pair $c(s)$ and $c(\overline{s})$ be of corresponding points of C and \overline{C} respectively. Then the curve \overline{C} is given by

$$\overline{c}(\overline{s}) = c(s) + \alpha(s) W_1(s), \quad (3.1)$$

where α is a function of s . Differentiating (3.1) with respect to s , we obtain

$$\frac{d\overline{s}}{ds} \frac{d\overline{c}(\overline{s})}{d\overline{s}} = c'(s) + \alpha'(s) W_1(s) + \alpha(s) W_1'(s).$$

Here the prime denotes the derivative with respect to s . By the Frenet equations, it holds that

$$\frac{d\overline{s}}{ds} \overline{L}(\overline{s}) = (1 - \alpha(s) k_1(s)) L(s) - \alpha(s) N(s) + \alpha'(s) W_1(s).$$

Since $\langle \overline{L}(\overline{s}), \overline{W}_1(\overline{s}) \rangle = 0$ and $\overline{W}_1(\overline{s}) = \mp W_1(s)$, we obtain $\alpha'(s) = 0$, that is, α is a constant function with value α (we can use the same letter without confusion). Thus (3.1) is rewritten as

$$\overline{c}(\overline{s}) = c(s) + \alpha W_1(s). \quad (3.2)$$

Differentiating (3.2) with respect to s , we obtain

$$\frac{d\overline{s}}{ds} \overline{L}(\overline{s}) = (1 - \alpha k_1(s)) L(s) - \alpha N(s). \quad (3.3)$$

From (3.3), we find

$$0 = -2\alpha(1 - \alpha k_1(s)).$$

Then we get

$$k_1(s) = \frac{1}{\alpha}.$$

So equation (3.3) becomes

$$\frac{d\overline{s}}{ds} \overline{L}(\overline{s}) = -\alpha N(s). \quad (3.4)$$

Differentiating both sides of (3.4) with respect to s , we arrive at

$$\frac{d^2\overline{s}}{ds^2} \overline{L}(\overline{s}) + \left(\frac{d\overline{s}}{ds}\right)^2 \overline{W}_1(\overline{s}) = -\alpha [k_1(s) W_1(s) + k_2(s) W_2(s)]. \quad (3.5)$$

Since $\langle W_2(s), \overline{W}_1(\overline{s}) \rangle = 0$, $\langle W_2(s), W_1(s) \rangle = 0$ and $\langle W_2(s), \overline{L}(\overline{s}) \rangle = 0$, we get

$$-\alpha k_2(s) = 0. \quad (3.6)$$

From the assumption in the hypothesis of the theorem, we have $k_2(s) \neq 0$. So from (3.6), we obtain $\alpha = 0$. Therefore (3.2) implies that \overline{C} coincides with C . This is a contradiction. This completes the proof.

4 (1,2)-Bertrand curves in R_1^4

Let C be a null Cartan curve in R_1^4 . We call W_1 the spacelike Cartan 1-normal vector field along C , and the spacelike Cartan 1-normal line of C at $c(s)$ is a line generated by $W_1(s)$ through $c(s)$. The spacelike Cartan (1,2)-normal plane of C at $c(s)$ is a plane spanned by $W_1(s)$ and $W_2(s)$ through $c(s)$.

Let C and \overline{C} be null Cartan curves in R_1^4 and $\varphi : s \rightarrow \overline{s}$ be a map such that each point $c(s)$ of C corresponds to the point $\overline{c}(\overline{s}) = \overline{c}(\varphi(s))$ of \overline{C} . Here s and \overline{s} are pseudo-arc parameters of C and \overline{C} respectively. If the Cartan (1,2)-normal plane at each point $c(s)$ of C coincides with the Cartan (1,2)-normal plane at corresponding point $\overline{c}(\overline{s}) = \overline{c}(\varphi(s))$ of \overline{C} and $\frac{d\varphi(s)}{ds} \neq 0$, then C is called (1,2)-Bertrand curve in R_1^4 and \overline{C} is called (1,2)-Bertrand mate of C .

Theorem 4.1 *Let C be a null Cartan curve in R_1^4 with curvature functions k_1, k_2 . Then C is a (1,2)-Bertrand curve iff there are constants α and β satisfying either*

$$(i) \quad \alpha = 0 \text{ and } \beta k_2(s) \neq 1$$

or

$$(ii) \quad \alpha \neq 0 \text{ and } \alpha k_1(s) + \beta k_2(s) = 1.$$

Proof: Now assume that C is a null Cartan (1,2)-Bertrand curve parametrized by the pseudo-arc parameter s in R_1^4 . The (1,2)-Bertrand mate \overline{C} is given by

$$\overline{c}(\overline{s}) = c(s) + \alpha(s) W_1(s) + \beta(s) W_2(s). \quad (4.1)$$

Here α and β are functions and \overline{s} is the pseudo-arc parameter of \overline{C} . Since the plane spanned by W_1 and W_2 coincides with the plane spanned by \overline{W}_1 and \overline{W}_2 , we can write

$$\begin{aligned} \overline{W}_1(\overline{s}) &= \cos \theta(s) W_1(s) + \sin \theta(s) W_2(s) \\ \overline{W}_2(\overline{s}) &= -\sin \theta(s) W_1(s) + \cos \theta(s) W_2(s) \end{aligned} \quad (4.2)$$

If we differentiate (4.1) with respect to s , then we get

$$\begin{aligned} \overline{L}(\overline{s}) \frac{d\overline{s}}{ds} &= [1 - \alpha(s) k_1(s) - \beta(s) k_2(s)] L(s) - \alpha(s) N(s) \\ &\quad + \alpha'(s) W_1(s) + \beta'(s) W_2(s). \end{aligned} \quad (4.3)$$

If we use (4.2) and (4.3) together, then we have

$$\begin{aligned} 0 &= \left\langle \overline{W}_1(\overline{s}), \overline{L}(\overline{s}) \frac{d\overline{s}}{ds} \right\rangle = \alpha'(s) \cos \theta(s) + \beta'(s) \sin \theta(s) \\ 0 &= \left\langle \overline{W}_2(\overline{s}), \overline{L}(\overline{s}) \frac{d\overline{s}}{ds} \right\rangle = -\alpha'(s) \sin \theta(s) + \beta'(s) \cos \theta(s). \end{aligned}$$

So we obtain $\alpha'(s) = \beta'(s) = 0$, that is α and β are constants. Then we can rewrite (4.3) as

$$\overline{L}(\overline{s}) \frac{d\overline{s}}{ds} = [1 - \alpha k_1(s) - \beta k_2(s)] L(s) - \alpha N(s). \quad (4.4)$$

From (4.4), we get

$$0 = \left\langle \overline{L}(\overline{s}) \frac{d\overline{s}}{ds}, \overline{L}(\overline{s}) \frac{d\overline{s}}{ds} \right\rangle = -2\alpha [1 - \alpha k_1(s) - \beta k_2(s)].$$

Here observe that it must be either

$$(i) \quad \alpha = 0 \text{ and } \beta k_2(s) \neq 1$$

or

$$(ii) \quad \alpha \neq 0 \text{ and } \alpha k_1(s) + \beta k_2(s) = 1.$$

Note that if both α and $[1 - \alpha(s)k_1(s) - \beta(s)k_2(s)]$ are zero in equation (4.4), this leads us to conclude that $\frac{d\bar{s}}{ds} = 0$. But we know from the definition of (1,2)-Bertrand curve the map $\varphi(s) = \bar{s}$ under which the points $c(s)$ of C and $\bar{c}(\bar{s})$ of \bar{C} correspond is a regular map. Therefore the case $\frac{d\bar{s}}{ds} = 0$ in (4.4) never happens. To characterize a (1,2)-Bertrand curve in R_1^4 , we delete the case $\frac{d\bar{s}}{ds} = \frac{d\varphi(s)}{ds} = 0$, and always think the position $\frac{d\bar{s}}{ds} \neq 0$.

Now we assume the contrary cases and prove that the curve C is a (1,2)-Bertrand curve.

$$(i) \quad \alpha = 0 \text{ and } \beta k_2(s) \neq 1.$$

Assume that C is a null Cartan curve in R_1^4 whose curvature functions $k_1(s)$ and $k_2(s)$ satisfy the relation $\beta k_2(s) \neq 1$ for some constant real numbers $\alpha = 0$ and β . Now we define a curve \bar{C} by

$$\bar{c}(s) = c(s) + \alpha W_1(s) + \beta W_2(s). \quad (4.5)$$

where s is the pseudo-arc parameter of C . Differentiating (4.5) with respect to s and using the Frenet equations, we obtain

$$\frac{d\bar{c}(s)}{ds} = [1 - \alpha k_1(s) - \beta k_2(s)] L(s) - \alpha N(s).$$

By using (i), we get

$$\frac{d\bar{c}(s)}{ds} = [1 - \beta k_2(s)] L(s). \quad (4.6)$$

Because of (i), the curve \bar{C} is a regular curve. Let us define a regular map $\varphi : s \rightarrow \bar{s}$ by

$$\bar{s} = \varphi(s) = \int_0^s \left\langle \frac{d^2\bar{c}(s)}{ds^2}, \frac{d^2\bar{c}(s)}{ds^2} \right\rangle^{\frac{1}{4}} ds,$$

where \bar{s} denotes the pseudo-arc parameter of \bar{C} . Then we obtain

$$\frac{d\bar{s}}{ds} = \frac{d\varphi(s)}{ds} = \sqrt{|1 - \beta k_2(s)|} > 0. \quad (4.7)$$

Thus the curve \bar{C} is rewritten as

$$\bar{c}(\bar{s}) = \bar{c}(\varphi(s)) = c(s) + \beta W_2(s). \quad (4.8)$$

If we differentiate (4.8) and use the Frenet equations for the null Cartan curve in R_1^4 , we have

$$\bar{L}(\bar{s}) \frac{d\bar{s}}{ds} = (1 - \beta k_2(s)) L(s). \quad (4.9)$$

We find the vector \bar{N} uniquely by using the formula in [6]. We get

$$\overline{N}(\overline{s})(1 - \beta k_2(s)) = \frac{d\overline{s}}{ds} N(s). \quad (4.10)$$

If we differentiate (4.9), then we obtain

$$\overline{W}_1(\overline{s}) \left(\frac{d\overline{s}}{ds} \right)^2 + \overline{L}(\overline{s}) \frac{d^2(\overline{s})}{ds^2} = (1 - \beta k_2(s))' L(s) + (1 - \beta k_2(s)) W_1(s). \quad (4.11)$$

From (4.11) we have

$$\left(\frac{d\overline{s}}{ds} \right)^2 = |1 - \beta k_2(s)| \neq 0.$$

Let us take $\left(\frac{d\overline{s}}{ds} \right)^2 = 1 - \beta k_2(s)$. Then equations (4.9) and (4.10) become

$$\overline{L}(\overline{s}) = \frac{d\overline{s}}{ds} L(s), \quad (4.12)$$

$$\overline{N}(\overline{s}) \frac{d\overline{s}}{ds} = N(s). \quad (4.13)$$

Now differentiating (4.12) with respect to s and using Frenet equations, we get

$$\overline{W}_1(\overline{s}) \frac{d\overline{s}}{ds} = \frac{d^2(\overline{s})}{ds^2} L(s) + \frac{d\overline{s}}{ds} W_1(s). \quad (4.14)$$

Using the equations (4.13) and (4.14) together, we obtain

$$0 = \left\langle \overline{N}(\overline{s}) \frac{d\overline{s}}{ds}, \overline{W}_1(\overline{s}) \frac{d\overline{s}}{ds} \right\rangle = \frac{d^2(\overline{s})}{ds^2}.$$

From here we can say that $\frac{d\overline{s}}{ds} = \ell_0$ is a constant. Then the equations (4.12) and (4.13) shrink to

$$\overline{L}(\overline{s}) = \ell_0 L(s), \quad (4.15)$$

$$\overline{N}(\overline{s}) = \frac{1}{\ell_0} N(s). \quad (4.16)$$

If we differentiate (4.15), then we get

$$\overline{W}_1(\overline{s}) \frac{d\overline{s}}{ds} = \ell_0 W_1(s).$$

Using the fact that $\frac{d\overline{s}}{ds} = \ell_0$, we have

$$\overline{W}_1(\overline{s}) = W_1(s). \quad (4.17)$$

If we differentiate (4.16), we have

$$[\bar{k}_1(\bar{s}) \bar{W}_1(\bar{s}) + \bar{k}_2(\bar{s}) \bar{W}_2(\bar{s})] \ell_0 = \frac{1}{\ell_0} [k_1(s) W_1(s) + k_2(s) W_2(s)]. \quad (4.18)$$

From (4.18), we get

$$\left[(\bar{k}_1(\bar{s}))^2 + (\bar{k}_2(\bar{s}))^2 \right] (\ell_0)^4 = [(k_1(s))^2 + (k_2(s))^2]. \quad (4.19)$$

Differentiating (4.17), we obtain

$$[-\bar{k}_1(\bar{s}) \bar{L}(\bar{s}) - \bar{N}(\bar{s})] \ell_0 = -k_1(s) L(s) - N(s). \quad (4.20)$$

By using (4.20), we get

$$\bar{k}_1(\bar{s}) = \frac{k_1(s)}{(\ell_0)^2}. \quad (4.21)$$

Now putting (4.21) into the equation (4.19), we have

$$|\bar{k}_2(\bar{s})| = \left| \frac{k_2(s)}{(\ell_0)^2} \right|.$$

Let us take

$$\bar{k}_2(\bar{s}) = \frac{k_2(s)}{(\ell_0)^2}. \quad (4.22)$$

If we use (4.17), (4.21) and (4.22) into (4.18), we obtain

$$\bar{W}_2 = W_2. \quad (4.23)$$

Observe that the coincidence of the plane spanned by W_1 and W_2 with the plane spanned by \bar{W}_1 and \bar{W}_2 is trivial. Therefore C is a (1,2)-Bertrand curve in R_1^4 .

(ii) $\alpha \neq 0$ and $\alpha k_1 + \beta k_2 = 1$.

Now we assume that C is a null Cartan curve whose curvature functions satisfy the equation $\alpha k_1 + \beta k_2 = 1$ for some constant real numbers $\alpha \neq 0$ and β . Think about a curve \bar{C} defined as

$$\bar{c}(s) = c(s) + \alpha W_1(s) + \beta W_2(s). \quad (4.24)$$

Differentiating (4.24) with respect to s and using the Frenet equations, we get

$$\frac{d\bar{c}(s)}{ds} = [1 - \alpha k_1(s) - \beta k_2(s)] L(s) - \alpha N(s).$$

By using (ii), we have

$$\frac{d\bar{c}(s)}{ds} = -\alpha N(s). \quad (4.25)$$

Because of (ii), the curve \bar{C} is a regular curve. Then we define a regular map $\varphi : s \rightarrow \bar{s}$ by

$$\bar{s} = \varphi(s) = \int_0^s \left\langle \frac{d^2 \bar{c}(s)}{ds^2}, \frac{d^2 \bar{c}(s)}{ds^2} \right\rangle^{\frac{1}{4}} ds,$$

where \bar{s} denotes the pseudo-arc parameter of \bar{C} . Then we obtain

$$\frac{d\bar{s}}{ds} = \frac{d\varphi(s)}{ds} = [\alpha^2 ((k_1)^2 + (k_2)^2)]^{\frac{1}{4}} > 0. \quad (4.26)$$

Now we can rewrite equation (4.24) as

$$\bar{c}(\bar{s}) = c(s) + \alpha W_1(s) + \beta W_2(s).$$

Differentiating this equation with respect to s , we get

$$\frac{d\bar{s}}{ds} \bar{L}(\bar{s}) = [1 - \alpha k_1(s) - \beta k_2(s)] L(s) - \alpha N(s).$$

Inserting (ii) in the above, we have

$$\frac{d\bar{s}}{ds} \bar{L}(\bar{s}) = -\alpha N(s). \quad (4.27)$$

Once again we use the formula in [6] to find \bar{N} . We find \bar{N} uniquely as

$$\bar{N}(\bar{s}) = -\frac{d\bar{s}}{ds} \frac{1}{\alpha} L(s) \quad (4.28)$$

Differentiating (4.27), we obtain

$$\frac{d^2 \bar{s}}{ds^2} \bar{L}(\bar{s}) + \left(\frac{d\bar{s}}{ds} \right)^2 \bar{W}_1(\bar{s}) = -\alpha [k_1(s) W_1(s) + k_2(s) W_2(s)]. \quad (4.29)$$

Using (4.28) and (4.29), we get

$$\begin{aligned} & \left\langle \bar{N}(\bar{s}), \left[\frac{d^2 \bar{s}}{ds^2} \bar{L}(\bar{s}) + \left(\frac{d\bar{s}}{ds} \right)^2 \bar{W}_1(\bar{s}) \right] \right\rangle \\ &= \left\langle -\frac{d\bar{s}}{ds} \frac{1}{\alpha} L(s), [-\alpha k_1(s) W_1(s) - \alpha k_2(s) W_2(s)] \right\rangle = 0. \end{aligned}$$

From the above fact, we get

$$\frac{d^2 \bar{s}}{ds^2} = 0.$$

So $\frac{d\bar{s}}{ds} = \ell_0$ is a constant. Using this in (4.29), we have

$$\bar{W}_1(\bar{s}) = -\frac{\alpha k_1}{(\ell_0)^2} W_1(s) - \frac{\alpha k_2}{(\ell_0)^2} W_2(s). \quad (4.30)$$

By using (4.30), we obtain

$$(\ell_0)^4 = \alpha^2 [(k_1)^2 + (k_2)^2]. \quad (4.31)$$

So we can write

$$\overline{W}_1(\overline{s}) = \cos \tau(s) W_1(s) + \sin \tau(s) W_2(s). \quad (4.32)$$

Differentiating (4.32), we arrive at

$$\begin{aligned} [-\overline{k}_1(\overline{s}) \overline{L}(\overline{s}) - \overline{N}(\overline{s})] \ell_0 &= \frac{d}{ds} [\cos \tau(s)] W_1(s) \\ &\quad + \cos \tau(s) [-k_1(s) L(s) - N(s)] \\ &\quad + \frac{d}{ds} [\sin \tau(s)] W_2(s) \\ &\quad + \sin \tau(s) [-k_2(s) L(s)]. \end{aligned}$$

Using (4.27) and (4.28) in the above, we have

$$\begin{aligned} \left[-\overline{k}_1(\overline{s}) \left(-\frac{\alpha}{\ell_0} N(s) \right) - \left(-\frac{\ell_0}{\alpha} L(s) \right) \right] \ell_0 &= \frac{d}{ds} [\cos \tau(s)] W_1(s) \\ &\quad + \cos \tau(s) [-k_1(s) L(s) - N(s)] \\ &\quad + \frac{d}{ds} [\sin \tau(s)] W_2(s) \\ &\quad + \sin \tau(s) [-k_2(s) L(s)]. \end{aligned} \quad (4.33)$$

Using the fact $\langle W_1(s), L(s) \rangle = \langle W_1(s), N(s) \rangle = \langle W_2(s), L(s) \rangle = \langle W_2(s), N(s) \rangle = 0$ in (4.33), we obtain

$$\frac{d}{ds} [\cos \tau(s)] = \frac{d}{ds} [\sin \tau(s)] = 0.$$

So the function $\tau(s)$ must be the constant $\tau(s_0)$. Using (4.33), we can also get

$$\overline{k}_1(\overline{s}) \alpha = -\cos \tau(s) = \frac{\alpha k_1(s)}{(\ell_0)^2}.$$

And from the above, we have

$$\overline{k}_1(\overline{s}) = \frac{k_1(s)}{(\ell_0)^2}. \quad (4.34)$$

Differentiating (4.28), we obtain

$$[\overline{k}_1(\overline{s}) \overline{W}_1(\overline{s}) + \overline{k}_2(\overline{s}) \overline{W}_2(\overline{s})] \ell_0 = -\frac{\ell_0}{\alpha} W_1(s). \quad (4.35)$$

From (4.35), we get

$$(\overline{k}_1(\overline{s}))^2 + (\overline{k}_2(\overline{s}))^2 = \frac{1}{\alpha^2}. \quad (4.36)$$

Using (4.31) and (4.34) into the equation (4.36), we get

$$|\overline{k}_2(\overline{s})| = \left| \frac{k_2(s)}{(\ell_0)^2} \right|.$$

Choosing,

$$\overline{k}_2(\overline{s}) = -\frac{k_2(s)}{(\ell_0)^2},$$

and putting it in (4.35), we obtain

$$\left\{ \frac{k_1(s)}{(\ell_0)^2} \left(-\frac{\alpha k_1(s)}{(\ell_0)^2} W_1(s) - \frac{\alpha k_2(s)}{(\ell_0)^2} W_2(s) \right) + \left(-\frac{k_2(s)}{(\ell_0)^2} \right) \overline{W}_2(\overline{s}) \right\} \ell_0 = -\frac{\ell_0}{\alpha} W_1(s).$$

Simplifying this equation, we arrive at

$$\overline{W}_2(\overline{s}) = \frac{\alpha k_2(s)}{(\ell_0)^2} W_1(s) - \frac{\alpha k_1(s)}{(\ell_0)^2} W_2(s) = -\sin \tau(s_0) W_1(s) + \cos \tau(s_0) W_2(s).$$

And it is trivial that the Cartan (1,2)-normal plane at each point $c(s)$ of C coincides with the Cartan (1,2)-normal plane at corresponding point $\overline{c}(\overline{s})$ of \overline{C} . Therefore C is a (1,2)-Bertrand curve in R_1^4 .

5 An example of (1,2)-Bertrand curve

Let a, b be nonzero constants such that $a \neq \mp b$, and let C be the curve in R_1^4 defined by

$$c(s) = \frac{1}{\sqrt{a^2 + b^2}} \left[\frac{1}{a} \sinh as, \frac{1}{a} \cosh as, \frac{1}{b} \sin bs, \frac{1}{b} \cos bs \right].$$

Then we get the Cartan frame and the curvature functions as follows:

$$\begin{aligned} L(s) &= \frac{1}{\sqrt{a^2 + b^2}} [\cosh as, \sinh as, \cos bs, -\sin bs], \\ W_1(s) &= \frac{1}{\sqrt{a^2 + b^2}} [a \sinh as, a \cosh as, -b \sin bs, -b \cos bs], \\ N(s) &= -\frac{\sqrt{a^2 + b^2}}{2} [\cosh as, \sinh as, -\cos bs, \sin bs], \\ W_2(s) &= \frac{1}{\sqrt{a^2 + b^2}} [b \sinh as, b \cosh as, a \sin bs, a \cos bs], \\ k_1 &= \frac{b^2 - a^2}{2}, \\ k_2 &= -ab. \end{aligned}$$

To enlighten the case (i), we take the constants real numbers α and β as follows:

$$\alpha = 0 \text{ and } \beta = \frac{1}{ab}.$$

Then

$$\alpha k_1 + \beta k_2 = -1 \neq 1$$

holds. Therefore C is a (1,2)-Bertrand curve in R_1^4 , and its Bertrand mate \overline{C} is given by

$$\overline{c}(\overline{s}) = \frac{2}{\sqrt{a^2 + b^2}} \left[\frac{1}{a} \sinh \left(a \frac{\overline{s}}{\sqrt{2}} \right), \frac{1}{a} \cosh \left(a \frac{\overline{s}}{\sqrt{2}} \right), \frac{1}{b} \sin \left(b \frac{\overline{s}}{\sqrt{2}} \right), \frac{1}{b} \cos \left(b \frac{\overline{s}}{\sqrt{2}} \right) \right],$$

where \overline{s} is the pseudo-arc parameter of \overline{C} , and a regular map $\varphi : s \rightarrow \overline{s}$ is given by

$$\overline{s} = \varphi(s) = \sqrt{2}s.$$

For the case (ii), we take real constants α and β as follows:

$$\alpha = \frac{1}{b^2 - a^2} \text{ and } \beta = -\frac{1}{2ab}.$$

Then it is trivial the following equation

$$\alpha k_1 + \beta k_2 = 1$$

holds. Therefore C is a (1,2)-Bertrand curve in R_1^4 , and its Bertrand mate is given by

$$\overline{c}(\overline{s}) = \frac{(\ell_0)^2}{\sqrt{a^2 + b^2}} \left[\frac{1}{a} \sinh \left(\frac{a}{\ell_0} \overline{s} \right), \frac{1}{a} \cosh \left(\frac{a}{\ell_0} \overline{s} \right), -\frac{1}{b} \sin \left(\frac{b}{\ell_0} \overline{s} \right), -\frac{1}{b} \cos \left(\frac{b}{\ell_0} \overline{s} \right) \right].$$

Here \overline{s} is the pseudo-arc parameter of \overline{C} , $\frac{d\overline{s}}{ds} = \ell_0$, and a regular map $\varphi : s \rightarrow \overline{s}$ is given by

$$\overline{s} = \varphi(s) = \sqrt{\frac{a^2 + b^2}{2(b^2 - a^2)}} s.$$

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